Representation Theory Summary

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Representations of Finite Groups

A ***representation*** of a finite group G on a $\mathbb{C}$ vector space V is a group homomorphism

$$G \to Aut(V)$$

The dimension of V is known as the ***degree*** of the representation.

A ***morphism*** of representations, (ρ_1, V_1) and (ρ_2, V_2), is a linear function $\phi : V_1 \to V_2$ such that

$$\forall g \in G \quad \rho_2(g)\phi = \phi\rho_1(g)$$

A ***subrepresentation*** of (ρ, V) is a vector subspace $W \leq V$ such that $\forall g \in G\ \rho(g)(W) \subseteq W$

Recall that for $W \leq V$ a projection of $V \to W$ is a linear map that restricted to W is the identity.

Lemma. *There is a bijection*

$$\{projections\ V \to W\} \leftrightarrow \{\ compliments\ of\ W\ in\ V\}$$

sending a projection to its kernel and a decomposition to the projection.

Lemma. *If $\rho : G \to GL(V)$ is a rep and $W \leq V$ is a subrep then there exists a complimentary subrep $W' \leq V$ such that $V = W \oplus W'$.*

Note that there is always a complimentary subspace but it might not be stable under the G action.

A representation is ***irreducible*** if it is non-trivial and has no non-trivial strict subreps.

Theorem. *Every representation decomposes into a direct sum of irreducible reps.*

Note that this is not true in general if we consider representations on non-$\mathbb{C}$ vector spaces, or infinite groups etc. This property is known as ***complete reducibility***.

Theorem (Schurs Lemma)**.** *If (ρ_1, V_1) and (ρ_2, V_2) are irreps and $f \in Hom_G(V_1, V_2)$ then $f = \lambda I$ for some $\lambda \in \mathbb{C}$. In particular f is either an iso or the zero map.*

Making New Reps

Let $(\rho, V), (\rho', V')$ be two representaitons of G.

Direct Sum

$$\tau : G \to GL(V \oplus V')$$

$$\tau(g)(v + v') = \rho(g)(v) + \rho'(g)(v')$$

Tensor If V and V' have a basis $\{v_i\}, \{v'_i\}$ respectively then $V \otimes V'$ has a basis $\{v_i \otimes v'_j\}$ and we get a new representation via

$$\tau : G \to GL(V \otimes V')$$

$$\tau(g)(v \otimes v') = \rho(g)(v) \otimes \rho'(g)(v')$$

Symmetric and Alternating Square A subcase is where we want to only use the first representation on $V \otimes V$. If we define

$$\theta : V \otimes V \to V \otimes V$$

$$v \otimes w \mapsto w \otimes v$$

Then we can write

$$V \otimes V = Sym^2(V) \oplus \Lambda^2(V)$$

where

$$Sym^2(V) = \{x \in V \otimes V : \theta(x) = x\}$$

$$\Lambda^2(V) = \{x \in V \otimes V : \theta(x) = -x\}$$

These spaces have respective bases $\{e_i \otimes e_j + e_j \otimes e_i\}$ and $\{e_i \otimes e_j - e_j \otimes e_i\}$. Then these are both subrepresentations on $V \otimes V$.

Dual Rep Given a vector space we can take its linear dual $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$.

$$\pi : G \to GL(V^*)$$

$$\pi(g)(f) = g.f$$

$$g.f(v) = f(\rho_{g^{-1}}(v))$$

Hom Rep

$$\pi : G \to GL(Hom_{\mathbb{C}}(V, V'))$$

$$\pi(g)(f)(v) = \rho'_g(f(\rho_{g^{-1}}(v)))$$

Characters

The ***character*** of a rep (ρ, V) is the map

$$\chi_\rho = \chi_V : G \to \mathbb{C}$$

$$\chi(g) = Tr(\rho(g))$$

or equally the sum of eigenvalues with multiplicity.

Lemma. *The character has the following properties*

- $\chi(1) = \dim V = \deg\rho$
- $\chi(g^{-1}) = \overline{\chi(g)}$ *the complex conjugate*
- $\chi(hgh^{-1}) = \chi(g)$*, it is a class function*
- $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$
- $\chi_{\rho\otimes\rho'} = \chi_\rho \cdot \chi_{\rho'}$
- $\chi_{\rho^*} = \overline{\chi_\rho}$
- $\chi_{Hom(V,V')} = \overline{\chi_V} \cdot \chi_{V'}$
- $\chi_{S^2V}(g) = \frac{1}{2}[\chi_V(g)^2 + \chi_V(g^2)]$
- $\chi_{\wedge^2V}(g) = \frac{1}{2}[\chi_V(g)^2 - \chi_V(g^2)]$

Lemma.

$$Hom_{\mathbb{C}}(V, V') \cong V^* \otimes V'$$

as representations.

A function $G \to \mathbb{C}$ is a ***class function*** if $\forall g, h \in G$ we have $f(hgh^{-1}) = f$, it is constant on conjugacy classes.

We define the following inner product on classfunction $\phi, \psi :: G \to \mathbb{C}$

$$\langle\phi, \psi\rangle = \frac{1}{|G|}\sum_{g\in G}\phi(g)\psi(g^{-1})$$

Theorem.

$$\langle\chi, \chi\rangle = 1$$

iff χ is an irreducible character.

Lemma. *If χ, χ' are two non-isomorphic irreducible characters then*

$$\langle \chi, \chi' \rangle = 0$$

Lemma. *If (ρ, V) is a G rep and V^G are the fixed points then*

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$$

Theorem. *V a rep of G and W_i an irrep of G. The number of W_i's contained in V as subrepresentations is then*

$$\langle \chi_V, \chi_{W_i} \rangle$$

Theorem. *Two representatinosn have the same character iff they are isomorphic.*

Counting Irreps

Theorem. *The number of irreducible representatinos of G is the same as the number of conjugacy classes.*

If W_i are all the irreps of G and they have respective dimensions n_i then

Lemma.

$$\sum_i n_i^2 = |G|$$

and

$$\sum_i n_i \chi_{W_i}(g) = 0 \quad \forall g \neq 1$$

Lemma. *If $g \in G$ we denote $c(g)$:= size of the conjugacy class of g. Then*

$$\sum_i \chi_i(g)\overline{\chi_i(g)} = \frac{|G|}{c(g)}$$

and for h not in the conjugacy class of g

$$\sum_i \chi_i(h)\overline{\chi_i(g)} = 0$$

Lemma. *The degrees of irreps of G divide $|G|/|\mathrm{cent}(G)|$*

Theorem. *$H \leq G$ normal subgroup and (ρ, V) an irrep of G then either*

- *$\rho|_H$ is isotypic (has only one irred component)*
- *OR; $\exists H \leq K < G$ and an irrep of K, σ, such that $\rho = \mathrm{Ind}_K^G \sigma$*

Lemma. *The degrees of irreps of G divide $|G|/|A|$ where A is any abelian normal subgroup of G.*

Canonical Decomposition

Ignoring

Regular Rep

The ***regular representation*** of G is the $\mathbb{C}$ vector space $span\{e_g : g \in G\}$ with the action that $h.e_g = e_{hg}$ which gets extended linearly to the rest of the vector space. This has the character

$$r_G(g) = \begin{cases} |G|, & g = 1 \\ 0, & else \end{cases}$$

Moreover it decomposes into

$$R_G = \oplus_i W_i^{\dim W_i}$$

where W_i runs over all the irreps of G.

Special Cases

Abelian Group

In an abelian group each conjugacy class has exactly one element.

Theorem. *G is abelian iff all irreducible reps have degree one.*

Lemma. *If A is an abelian subgroup of G then every irreducible rep of G has degree* $\leq \frac{|G|}{|A|}$

Products of Groups

Let G_1, G_2 be two groups with respective representations $(\rho_1, V_1), (\rho_2, V_2)$. We get a representation

$$\rho_1 \otimes \rho_2 : G_1 \times G_2 \to GL(V_1 \otimes V_2)$$

$$(g_1, g_2) \mapsto \rho_1(g_1) \otimes \rho_2(g_2)$$

This has character the product of the two other characters as before.

Theorem.

- *If ρ_i are both irreps then $\rho_1 \otimes \rho_2$ is irrep*
- *Every irrep of $G_1 \times G_2$ is iso to something of this form.*

Semi-Direct Product

Let $G = H \ltimes A$ where A is normal and abelian. All iirps of A are one dimensional, hence they form a group namely $X = \text{Hom}(A, \mathbb{C}^*)$. There is a G action on X via

$$g.\chi(a) = \chi(g^{-1}ag)$$

Then for each character we get a subgroup of H,

$$H_i \quad := \quad \{h \in H : h.\chi_i = \chi_i\}$$

Note that if χ_i is in the same H orbit as χ_j then $H_i = H_j$. Let $G_i = H_i \ltimes A$ the characters of A extend to characters of G_i by just ignoring the H_i, $\chi_i(ha) = \chi_i(a)$. Irreps of H_i extend in the same way to irreps of G_i. Then if ρ is such an irrep of G_i

- $\text{Ind}_{G_i}^{G}(\chi_i \otimes \rho)$ is irreducible
- Every irrep of G arrises in this way.

Solvable and Sylow Groups

Recall that a ***solvable*** group G has a finite derived series i.e.

$$\{1\} \leq G_0 \leq \cdots \leq G_n = G$$

such that $G_{i-1} \leq G_i$ normal and G_i/G_{i-1} is abelian.

A ***supersolvable*** group moreover has that G_i/G_{i-1} is cyclic. Finally a ***nilpotent*** group is one that is solvable and $G_i/G_{i-1} \leq Cent(G_i/G_{i-1})$.

Lemma. *Nilpotent* $\implies$ *Supersolvable* $\implies$ *Solvable*

A ***p-group*** is a group whose order is a power of p, for p prime.

Theorem. *Every p-group is nilpotent*

Lemma. *If a p-group G acts on a finite set X then*

$$|X| \equiv |X^G| \; (mod\ p)$$

Lemma. *Let V be a non-zero k-vector space, where characteristic of k is p. Let (ρ, V) be a rep of G. If G is a p-group then there exists a $v \in V - \{0\}$ that is fixed by $\rho(g)$ for all $g \in G$ i.e.*

$$\forall g \in G \quad \rho(g)v = v$$

Theorem. *The only irreducible rep of a p-group in characteristic p is the trivial rep.*

Recall that for a group G a Sylow-p subgroup is a maximal p-subgroup.

Theorem. *If p is prime and $|G| = mp^n$ for some m coprime to p. Then*

- *There exists a Sylow p-subgroup (of order n)*
- *All Sylow-p subgroups are conjugate i.e. For any two Sylow-p subgroups P, Q there exists a $g \in G$ such that*

$$gPg^{-1} = Q$$

- *Each p-subgroup of G is contained in a Sylow-p subgroup*

Lemma. *G non-abelian and supersolvable then there is a normal abelian subgroup that is not contained in the center of G.*

Lemma. *Every irreducible representation of a supersolvable group is induced by a degree one represntation of a subgroup.*

Induction & Restriction

Let $H \leq G$ a subgroup and $R = G/H$ a collection of representatives of cosets. Let W be a $\mathbb{C}[H]$ module. We define the ***induction*** of W to G to be the representation

$$\operatorname{Ind}_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

with $\mathbb{C}[G]$ action

$$g.(x \otimes w) = (gx) \otimes w$$

which has the following properties

- $\dim \operatorname{Ind}_H^G W = [G : H] \dim W$
- $\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{s \in G, s^{-1}gs \in H} \chi_W(s^{-1}gs)$
- $\operatorname{Ind}_H^G \cong \operatorname{Hom}_H(\mathbb{C}[G], W)$ as representations.
- $\operatorname{Hom}_H(W, \operatorname{Res}_H^G E) \cong \operatorname{Hom}_G(\operatorname{Ind}_H^G W, E)$
- $V \otimes_{\mathbb{C}} \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G(\operatorname{Res}_H^G V \otimes_{\mathbb{C}} W)$
- If $H \leq K \leq G$ then $\operatorname{Ind}_K^G \operatorname{Ind}_H^K W \cong \operatorname{Ind}_H^G W$

Induction is a functor: If $f : H \to \mathbb{C}$ is a class function then $\operatorname{Ind}_H^G f : G \to \mathbb{C}$ definied by

$$\operatorname{Ind}_H^G f(g) = \frac{1}{|H|} \sum_{s \in G, s^{-1}gs \in H} f(s^{-1}gs)$$

is a calss function. Moreover if f is a character of W then

$$\operatorname{Ind}_H^G \chi_W = \chi_{\operatorname{Ind}_H^G W}$$

Lemma.
$$\dim \operatorname{Hom}_H(V_1, V_2) = \langle \chi_{V_1}, \chi_{V_2} \rangle$$

Lemma.
$$\langle \psi, \operatorname{Res} \phi \rangle_H = \langle \operatorname{Ind}_H^G \psi, \phi \rangle_G$$

Lemma. *W irrep of H, E irrep of G. Then the number of times that W appears in* $\operatorname{Res}_H^G E$ *is the number of times E occurs in* $\operatorname{Ind}_H^G W$

Ignoring a lot of lecture 10 and 11

Restriction

Let $H \leq G$ and $K \leq G$ be two subgroups. We are going to induce one of the subgroups and restrict down to the other. Let (ρ, W) be a H rep. Consider the double cosets $K \setminus G/H := \{KgH : g \in G\}$ where $KgH = \{kgh : k \in K, h \in H\}$. For $s \in G$ we define $H_s := sHs^{-1} \cap K$ and $\rho^s : H_s \to GL(W)$ sending $x \mapsto \rho(s^{-1}xs)$.

Theorem.

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G(W) = \bigoplus_{s \in K \setminus G/H} \operatorname{Ind}_{H_s}^K(\rho^s)$$

s some representatives

Mackey's Irreducibility

Let $H \leq G$ and $s \in G$, we define $H_s := sHs^{-1} \cap H$. If $\rho : H \to GL(W)$ is a rep then so is $\operatorname{Res}_{H_s}^H \rho$ and

$$\rho^s : H_s \to GL(W)$$

$$x \mapsto \rho(s^{-1}xs)$$

Theorem. $\operatorname{Ind}_H^G W$ *is irreducible iff W is irreducible and* $\forall s \in G - H$

$$\langle \rho^s, \operatorname{Res}_{H_s}^H \rangle = 0$$

Lemma. *If H is normal in G then* $\operatorname{Ind}_H^G W$ *is irreducible iff W is irreducible and* $\forall s \in G - H \; \rho \not\cong \rho^s$

Group Algebra

We now work with G a finite group and K a commutative ring of characteristic zero. We make $K[G] := span_K\{g : g \in G\} \cong K^{|G|}$ the formal span of G as a K module into a ring and therefore a K algebra by defining multiplication as

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{g,h\in G} a_g b_h gh$$

This is called the ***group algebra***

If k is a field, V is a k vector space and $\rho : G \to GL(V)$ is a rep then V can be made into a left $k[G]$ module via

$$\left(\sum_{g\in G} a_g g\right).v = \sum_{g\in G} a_g \rho(g)(v)$$

If V is a left $k[G]$ module then the following is a rep

$$g \mapsto (v \mapsto g.v)$$

Theorem. *In characteristic 0 we have complete reducibility. Every $k[G]$ module decomposes into the direct sum of irreducible submodules.*

Decomp of $\mathbb{C}$ [G]

Consider the irreps of G, (ρ_i, W_i).

Theorem.

$$\mathbb{C}[G] \cong \prod_i M_{\dim W_i}(\mathbb{C})$$

$$g \mapsto (\rho_1(g), ..., \rho_m(g))$$

extended linearly to all of $\mathbb{C}[G]$

Lemma. *Let $(u_1, ..., u_m) \in \prod_i End(W_i)$ and $u = \sum_{g\in G} u(g)g$ its preimage under the above iso. Then*

$$u(g) = \frac{1}{|G|}\sum_i \dim(W_i)Tr(\rho_i(g^{-1})u_i)$$

Center

If c is a conjugacy class of G then we denote $z_c = \sum_{g\in c} g$.

Lemma. *$\{z_c\}_c$ forms a basis of the center of $\mathbb{C}[G]$.*

Theorem. *From the decomposition of $\mathbb{C}[G]$ we have homomorphisms (that when collected give an isomorphism) $\rho_i : \mathbb{C}[G] \to End(W_i) = Mat_{\dim W_i}(\mathbb{C})$ which restricts to the center $\omega_i = \rho_i|_{Cent(\mathbb{C}[G])}$ for $i = 1, ..., k$.*

$$(\omega_i)_i : cent(\mathbb{C}[G]) \to \mathbb{C}^k$$

Is an isomorphism. Explicitly

$$\omega_i\left(\sum u(g)g\right) = \frac{1}{\dim W_i}\sum_{g\in G} u(g)\chi_i(g)$$

Ignoring algebraic integer stuff... Lecture 15

Symmetric Groups

Conjugacy classes of S_n are in bijection with partitions of n.

Young Subgroups

Fix a tableau t_λ of shape λ. Then the young subgroups associated are

$$P = P_{t_\lambda} = \{g \in S_n : g \text{ preserves each row}\}$$

$$Q = Q_{t_\lambda} = \{g \in S_n : g \text{ preserves each column}\}$$

Note that S_n acts on the tableux by permuting the numbers. By preseve the rows / columns we are saying the same numbers are in there, we dont care about the order (otherwise both groups would be trivial).

Lemma. *If $\lambda = \lambda_1 + \cdots + \lambda_m$ with transpose $\lambda' = \mu_1 + \cdots + \mu_{m'}$ then the young subgroups are*

$$P \cong S_{\lambda_1} \times \cdots \times S_{\lambda_m}$$

$$Q \cong S_{\mu_1} \times \cdots \times S_{\mu_{m'}}$$

There are three distinguished elements of $\mathbb{C}[S_n]$ associated to the young subgroups

$$a_\lambda = \sum_{g\in P} e_g, \quad b_\lambda = \sum_{g\in Q} sgn(g) e_g, \quad c_\lambda = a_\lambda b_\lambda$$

Classification of Irreps

Theorem.

- *Some scaler multiple of c_λ is idempotent i.e.*

$$\exists n_\lambda \in \mathbb{C}, \quad c_\lambda^2 = n_\lambda c_\lambda$$

- *For every λ, $\mathbb{C}[S_n]c_\lambda$ is an irreducible representation of S_n*
- *Every irreducible representation of S_n is given by $\mathbb{C}[S_n]c_\lambda$ for some λ*

We will need several lemmas to prove this result which we now develop.

First notice that $P \cap Q = \{1\}$, thus $\forall g \in S_n$ we can write it in *at most one way* as $g = pq$ where $p \in P, q \in Q$ (it might be not expressable in this form). Applying these to our distinguished element

$$c_\lambda = \sum_{pq\in PQ} sgn(q) e_{pq}$$

Ignoring lectures 22, 23; have the lemmas for the proof etc

Applications

Lemma. *For any λ we have that $c_\lambda^2 = n_\lambda c_\lambda$*

$$n_\lambda = \frac{n!}{\dim(\mathbb{C}[S_n]c_\lambda)}$$

5.3.1 Frobenius Formula

We set up some notation. Let $V_\lambda = \mathbb{C}[S_n]c_\lambda$ and χ_λ the associated character. We can write a partition multiplicatively

$$\lambda = \lambda_1 + \cdots + \lambda_m \rightsquigarrow \lambda_1 \cdots \lambda_m$$

which if we have repeated entries we can collpse to be something of the form

$$\lambda_i = n^{i_n}(n-1)^{i_{n-1}} \cdots 1^{i_1}$$

so to a tuple of non-negative integers $i = (i_1, \cdots, i_n)$ we can associate a partition λ_i above.

Fix a $k \geq$ the number of rows in λ_i and let

$$p_j(x) = x_1^j + \cdots x_k^j$$

and

$$\Delta(x) = \prod_{1\leq i<j\leq k} (x_i - x_j)$$

Finally if $f(x) = f(x_1, ..., x_k)$ is a formal power series and $(\ell_1, ..., \ell_k) \in \mathbb{Z}_{\geq 0}^k$ then we denote

$$[f(x)]_{(\ell_1,...,\ell_k)} = \text{ the coefficient of } x_1^{\ell_1} \cdots x_k^{\ell_k} \text{ in } f$$

Theorem.

$$\chi_\lambda(C_i) = \left[\Delta(x) \prod_{j=1}^n p_j(x)^{i_j}\right]_{(\ell_1,...,\ell_k)}$$

As a corrolory we know that

$$\dim V_\lambda = \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{i<j} (\ell_i - \ell_j) = \frac{n!}{\prod \text{ hook lengths in } \lambda}$$

and this is further independent of our choice of k.

Lemma. *The dimension of $\mathbb{C}[S_n]c_\lambda$ is equal to the number of tableux on λ such that the rows and columns are increasing*

Slide 153 remarks, Schur-Weyl duality

Alternating Groups

$A_n \leq S_n$ is the commutator subgroup of S_n, it has index two. For a general subgroup of index two $H \leq G$ we have the permutation representation of $G \circlearrowright G/H = \{1, r\}$ which decomposes into a direct sum $\mathbb{C}_{triv} \oplus \mathbb{C}_{non-triv}$

Lemma. *Let V be an irrep of G, and let $W = \mathrm{Res}_H^G V$ where $H \leq G$ an index two subgroup. Let $V' = V \otimes \mathbb{C}_{non-triv}$ for the permutation representation. Then one of the following holds*

- *$V \ncong V'$: Then W is irreducible and $\mathrm{Ind}_H^G W \cong V \oplus V'$*
- *$V \cong V'$: Then $W \cong W' \oplus W''$ for two non-isomorphic irreps W', W''. Moreover*

$$V \cong \mathrm{Ind}_H^G W' = \mathrm{Ind}_H^G W''$$

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General Representations of Algebras

We set k to be an algebraically closed field. A k ***algebra*** is a k vector space with a bilinear multiplication. A left A-***module*** is a k-vector space V with a homomorphism $\rho : A \to End_k(V)$, a linear map preserving multiplication and unit.

A ***submodule*** is a subspace $U \leq V$ such that $\rho(a)U \leq U$ for all $a \in A$.

A non-zero rep V of A is ***irreducible*** if its only subrepresentations are V and $\{0\}$. It is ***indecomposable*** if it cannot be written as a direct sum of two non-zero sub representations.

If V_1, V_2 are reps of A then $V_1 \oplus V_2$ is too via

$$a(v_1 + v_2) = av_1 + av_2$$

A ***homomorphism*** is a map $\phi : V_1 \to V_2$ such that $\phi(a.v) = a.\phi(v)$.

Theorem (Schur's Lemma). *If V_1, V_2 are A reps and $\phi : V_1 \to V_2$ is a non-zero hom then*

- *If V_1 is irreducible, then ϕ is injective*
- *If V_2 is irreducible, then ϕ is surjevtive*
- *If V_1, V_2 is irreducible, then ϕ is an isomorphism*

moreover if $V_1 = V_2$ and ϕ is an iso then it acts by a scalar.

Lemma. *If A is commutative algebra then every finite dim rep of A is one dimensional*

A left (right) ***ideal*** of a k-algebra A is a subspace $I \leq A$ that is closed under left (right) multiplication by A. We call a space that is both a left and right idea a ***two sided ideal***.

An algebra A is ***simple*** if 0 and A are its only two sided ideals.

A representation of an algebra is called ***faithful*** if it is injective.

A rep of A is called ***semi-simple*** if it is the direct sum of irreducible representations.

Lemma. *V_i finite dimensional irreps of A. If*

$$W \leq V = \sum_i V_i^{n_i}$$

then for some $r_i \leq n_i$

$$W \cong \oplus_i V_i^{r_i}$$

and there is a morphism of representations $\phi : W \to V$ given on components $V_i^{r_i} \to V_i^{n_i}$ by multiplying on the right by X_i an $r_i \times n_i$ matrix with linearly independent rows.

Lemma. *If $V = \oplus_{i \in I} V_i$ where each V_i is irreducible and $f : V \to U$ is surjective then there exists a subset $J \subseteq I$ such that f maps $\oplus_{i \in J} V_i$ isomorphically onto U.*

Lemma. *V irrep of finite dimension and $v_1, ..., v_n \in V$ linearly independent. Then for every other collection $w_1, ..., w_n \in V$ there exists an $a \in A$ such that for every i*

$$av_i = w_i$$

Theorem (Density Theorem). *If (ρ, V) is a finite dimensional irrep of A then ρ is a surjection.*

Given a rep of A, V then $V^* := Hom_k(V, k)$ is a right A module via

$$(f.a)(v) = f(a.v)$$

Lemma. *If $A = \prod_i Mat_{d_i}(k)$ then the irreps of A are $V_1 = k^{d_1}, ..., V_r = k^{d_r}$, moreover any finite dimensional rep of A is isomorphic to $\oplus V_i^{n_i}$*

Filtrations

A finite ***filtration*** of V is a sequence of subreps $0 = V_0 < V_1 < \cdots < V_n = V$

Lemma. *Every finite dimensional rep V of A admits a finite filtration such that V_i/V_{i-1} is irreducible for all i.*

Such a filtration is called a ***composition series***.

Theorem. *Any two composition series of V are the same length and the quotients are isomorphic (up to reordering).*

The collection of irreps $\{V_i/V_{i-1} : i = 1, ..., n\}$ is called the ***Jordan-Holder series*** of V.

Theorem (Krull-Schmidt). *Any finite dimensional rep V of A can be uniqueqly decomposed into a direct sum of indecomposable subreps*

Lemma. *If W is a finite dimensional indecomposable rep of A then a hom $W \to W$ is either nilpotent or an isomorphism. Moreover the sum of nilpotent maps is nilpotent.*

If $\theta : W \to W$ is a homomorphism of A reps then for $\lambda \in k$ we have $W_\lambda = \{w \in W : (\theta - \lambda)^n w = 0 \text{ some n}\}$, this is the ***generalised eigenspace***.

Finite Dimensional

Let A be a finite dimensional k algebra. The ***radical*** of A is the set

$$Rad(A) = \{a \in A : aV_i == \quad \forall V_i \text{ irreducible reps}\}$$

Lemma. *Rad(A) is a two sided ideal.*

Lemma. *Let I be a nilpotent two sided ideal, then it is contained in the radical. Moreover the radical is the largest nilpotent two sided ideal.*

Theorem. *A has only finitely many irreps up to isomorphism. Moreover*

$$A/Rad(A) \cong \prod_i End_k(V_i)$$

where V_i are the irreps of A.

Lemma. *For the irreps of A, V_i we have that*

$$\sum_i (\dim V_i)^2 \leq \dim A$$

A finite dimensional algebra is ***semisimple*** if its radical is zero.

Theorem. *For a finite dimensional algebra A the following are equivilent*

- *A is semisimple*
- $\sum_i (\dim V_i)^2 = \dim A$
- $A \cong \prod_i Mat_{d_i}(k)$
- *Every finite dim rep of A is completely reducible*
- *Every subrep of a finite dim rep of A admits a complementary subrep.*
- *A is completely reducible as an A module.*

Characters

Characters are defined in the same way.

Lemma. *Characters of irreps of A are linearly independent.*

Lemma. *If A is finite dimensional semisimple algebra then the characters form a basis of* $(A/[A, A])^*$

Weyl Algebra

The ***Weyl algebra*** is

$$k[x, y]/\langle yx - xy - 1\rangle$$

Theorem. *The Weyl algebra has a basis* $\{x^i y^j : i, j \geq 0\}$

Tensor Product

If A, B are k-algebras then we can make the tensor

$$A \otimes_k B$$

with the multiplication

$$(a \otimes b)(\alpha \otimes \beta) = a\alpha \otimes b\beta$$

If V is a rep of A and W is a rep of B then $V \otimes_k W$ is a rep of $A \otimes_k B$ via the action

$$(a \otimes b)(v \otimes w) = av \otimes bw$$

Lemma. *V irrep of A, W irrep of B then* $V \otimes W$ *is irrep of* $A \otimes B$*. Moreover every irrep of* $A \otimes B$ *is of this form.*

Theorem. *A, B subalgebras of End(E), where E is a finite dimensional k-vector space, such that A is semisimple and* $B = End_A(E)$*. Then*

- $A = End_B(E)$
- *B is semi-simple*
- *As a rep of* $A \otimes B$ *E decomposes as a direct sum over the tensor of irreps of A and B*

$$E = \bigoplus_i V_i \otimes W_i$$

for some ordering on the irreps of A and B.

Lecture 34, enveloping algebra

Structure of Finite Dimensional Algebras

Let A be a k-algebra and $I \leq A$ a nilpotent two sided ideal. Recall that an element of an algebra is called ***idempotent*** when $e^2 = e$.

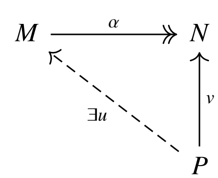
Lemma. *Let* $e \in A/I$ *be idempotent then there exists idempotent* $f \in A$ *such that* $\pi(f) = e$*, moreover this lift is unique up to conjagacy by an element of* $1 + I$*.*

A ***complete system of orthogonal idempotents*** in a k-algebra A is a collection of idempotents $e_1, ..., e_m \in A$ such that $e_i e_j = 0$ for each $i \neq j$ and $e_1 + \cdots + e_m = 1$.

Lemma. *If* $e_1, ..., e_m$ *is a complete system of idempotents of* A/I *then there exists a complete system of idempotents in A,* $f_1, ..., f_m$ *such that* $\pi(f_i) = e_i$*.*

Theorem. *If P, M, N are representaitons of A (Left A modules) then the following are equivilent*

- *If* $\alpha : M \to N$ *is a surjection and* $P \to N$ *then there exists a* $u : P \to M$ *such that* $\alpha \circ u = v$*.*



- *If* $\alpha : M \to N$ *is a surjection then there exists a* $u : P \to M$ *such that* $\alpha \circ u = id$ *i.e.* α *splits.*
- *There exists an A module Q such that* $P \oplus Q$ *is a free A module*
- *The functor* $\mathrm{Hom}_A(P, -)$ *is exact*

A module satisfying one of the above conditions is called ***projective***.

Theorem. *A finite dimensional with irreps* $M_1, ..., M_n$ *we have*

- *For each* $i = 1, ..., n$ *there exists a unique indecomposable finitely generated projective A module* P_i *such that* $\dim \mathrm{Hom}_A(P_i, M_j) = \delta_{ij}$
- $A \cong \oplus_i P_i^{\dim M_i}$
- *Any indecomposable finitely generated projective A module is isomorphic to* P_i *for some i*

The P_i is called the ***projective cover*** of M_i.

Lemma. *If N is a finite dimensional rep of A then* $\dim \mathrm{Hom}_A(P_i, N) = [N : M_i]$ *which is the multiplicity of the occurence of* M_i *in the Jordan-Holder series.*

If A has irreps M_i with projective covers P_i then the matrix

$$C_{ij} := \dim \mathrm{Hom}_A(P_i, P_j) = [P_j : M_i]$$

is called the ***Cartan matrix*** of A.